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FIRST INTEGRALS IN THE PROBLEM OF THE MOTION OF A HEAVY RIGID BODY ABOUT A FIXED POINT UNDER UNILATERAL CONSTRAINT†

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The problem of the existence of integrable cases of the Euler and Lagrange types, and also particular integrals of the Hess and Appel'rot type without additional assumptions on the value of the area integral, is considered for the problem of the motion of a heavy rigid body about a fixed point with constraints on the angle between the rising vertical and a vector fixed in the body.

It was shown in [1, 2], that at the zero level the area integral of the problem of the motion of a heavy rigid body about a fixed point with a constraint on the change in the nutation angle, reduces to the problem of the motion of a point in a special spherical billiards. It was established in [1, 2], that when the Kovalevskoya and Goryachev–Chaplygin conditions are satisfied, both problems also remain integrable when such a constraint is imposed.

1. FORMULATION OF THE PROBLEM

Consider the motion of a heavy rigid body about a fixed point O. Suppose $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity vector of the body, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is the vector of the rising vertical, $\mathbf{c} = (c_1, c_2, c_3)$ is a vector connected with the fixed point O and the centre of mass of the body – the point C, m is the mass of the body and g is the acceleration due to gravity. Then, as usual, the equations of motion in the $Ox_1x_2x_3$ axes, directed along the principal axes of the inertia tensor of the body $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$ relative to the fixed point, can be written in the form

$$\mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} + mg\boldsymbol{\gamma} \times \mathbf{c}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega} \tag{1.1}$$

These equations allow of the energy integral

$$\mathcal{T}_0 = \frac{1}{2}(\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\omega}) + mg(\mathbf{c}, \boldsymbol{\gamma}) \tag{1.2}$$

the area integral

$$\mathcal{T}_\psi = (\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\gamma}) \tag{1.3}$$

and the geometrical integral

$$\mathcal{T}_g = (\boldsymbol{\gamma}, \boldsymbol{\gamma}) \tag{1.4}$$

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2. THE EQUATIONS OF MOTION WITH A CONSTRAINT

Suppose $\mathbf{e} = (e_1, e_2, e_3)$ is the unit vector fixed in the body. We impose the following constraint on the system

$$f = (\boldsymbol{\gamma}, \mathbf{e}) - \gamma_e \geq 0 \tag{2.1}$$

Physically this means that the angle between the rising vertical and the vector $\mathbf{e}$ does not exceed an amount $\arccos\gamma_e$. To realize this constraint we can use, for example, a tether of length $\sqrt{2-2\gamma_e}$, connecting the ends of the vectors $\mathbf{e}$ and $\boldsymbol{\gamma}$.

According to the general theory of the motion of rigid bodies when there are impact interactions [3], the values of the angular velocity of the body before the impact ($\boldsymbol{\omega}$) and after the impact ($\boldsymbol{\omega}'$) are connected by the relation

$$\mathbf{I}(\boldsymbol{\omega}' - \boldsymbol{\omega}) = R\mathbf{f}, \quad \mathbf{f} = \boldsymbol{\gamma} \times \mathbf{e} \tag{2.2}$$

where, since the impact is assumed to be absolutely elastic, the value of R can be found from the condition of the conservation of kinetic energy (the potential energy remains unchanged during the impact time)

$$(\mathbf{I}\boldsymbol{\omega}', \boldsymbol{\omega}') = (\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\omega}) \tag{2.3}$$

Assertion 1. The quantities

$$\mathcal{T}_e = (\mathbf{I}\boldsymbol{\omega}, \mathbf{e}), \quad \mathcal{T}_\psi = (\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\gamma}) \tag{2.4}$$

remain constant during the impact time.

The proof reduces to multiplying the left- and right-hand sides of Eq. (2.2) scalarly by the vectors $\mathbf{e}$ and $\boldsymbol{\gamma}$ respectively.

Hence, in particular, the projection of the angular momentum vector onto the vertical remains unchanged not only during the continuous motion but also during the time of the impact, i.e. over the whole time of motion.

Definition of the reaction. To determine the reaction R we multiply the left- and right-hand sides of Eq. (2.2) scalarly by $\boldsymbol{\omega} + \boldsymbol{\omega}'$. We have

$$(\mathbf{I}\boldsymbol{\omega}', \boldsymbol{\omega}') - (\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\omega}) = R(\mathbf{f}, \boldsymbol{\omega}' + \boldsymbol{\omega}) = 0$$

Then, if $R \neq 0$, we have

$$(\mathbf{f}, \boldsymbol{\omega}' + \boldsymbol{\omega}) = 0 \tag{2.5}$$

At the instant of reaching the constraint $\mathbf{f} \neq 0$, otherwise $(\boldsymbol{\gamma}, \mathbf{e}) = \pm 1$. Then, expressing the quantity $\boldsymbol{\omega}'$ from (2.2) we have

$$\boldsymbol{\omega}' = \boldsymbol{\omega} + R \cdot \mathbf{I}^{-1}\mathbf{f} \tag{2.6}$$

Substituting this expression into Eq. (2.5) we obtain

$$R = -2\frac{(\boldsymbol{\omega}, \mathbf{f})}{(\mathbf{I}^{-1}\mathbf{f}, \mathbf{f})} \tag{2.7}$$

3. CASES WHEN ADDITIONAL INTEGRALS EXIST

The Euler case. As is well known, in the Euler case the square of the angular momentum vector $\mathcal{T}_K = (\mathbf{I}\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega})$ remains unchanged; multiplying the left- and right-hand sides of Eq. (2.2) scalarly by $\mathbf{I}(\boldsymbol{\omega}' + \boldsymbol{\omega})$, we have from the condition for the square of the angular momentum vector to be constant,

$$(\mathbf{I}\boldsymbol{\omega}', \mathbf{I}\boldsymbol{\omega}') - (\mathbf{I}\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega}) = R(\boldsymbol{\gamma} \times \mathbf{e}, \mathbf{I}(\boldsymbol{\omega}' + \boldsymbol{\omega})) = 0$$

whence it follows that

$$R = -2\frac{(\mathbf{I}\boldsymbol{\omega}, \boldsymbol{\gamma} \times \mathbf{e})}{(\boldsymbol{\gamma} \times \mathbf{e}, \boldsymbol{\gamma} \times \mathbf{e})} \tag{3.1}$$

In the general case, the right-hand sides of expressions (2.7) and (3.1) are not the same, and of course, in the general case for absolutely elastic impacts, the square of the angular momentum vector is not conserved. This indicates that there is no integrable case of the Euler form.

The Lagrange case. In the Lagrange case certain conditions for the existence of an additional integral are satisfied, for example, $I_1 = I_2$ and $c_1 = c_2 = 0$. Then, if the vectors **c** and **e** are collinear, by virtue of relation (2.6) we have

$$\omega_3' = \omega_3$$

and all the first integrals are also conserved in the case when a unilateral constraint is imposed.

The Hess–Appel'rot case. Suppose, to fix our ideas, $I_1 > I_2 > I_3$. We will put $\Delta_{ij} = \sqrt{I_i^{-1} - I_j^{-1}}$. As is well known, in the case of motion without a constraint, when the conditions

$$\mathbf{a}_\varepsilon = (a_1, a_2, \varepsilon a_3): a_1 = \Delta_{21}, \quad a_2 = 0, \quad a_3 = \Delta_{32}, \quad \varepsilon = \pm 1 \tag{3.2}$$

$$\mathbf{c} = \lambda \mathbf{a}_\varepsilon, \quad \lambda = \text{const} \tag{3.3}$$

are satisfied, a particular integral exists, which has the form

$$F_\varepsilon = a_1 I_1 \omega_1 + \varepsilon a_3 I_3 \omega_3 = 0, \tag{3.4}$$

Then, by virtue of Assertion 1, if the vectors **a** and **e** are collinear, particular integrals F_ε also exist for a system under unilateral constraint (2.1).

Remark. The results presented above, which touch on the existence of first integrals, remain true when two constraints of the form

$$\gamma_e^- \le (\boldsymbol{\gamma}, \mathbf{e}) \le \gamma_e^+ \tag{3.5}$$

are imposed on the system. Physically this means that the angle between the rising vertical and the vector **e** is enclosed between the angles $\arccos\gamma_e^+$ and $\arccos\gamma_e^-$. To realise this constraint, one can use, for example, a tether of length $\sqrt{2 - 2\gamma_e^-}$, connecting the ends of the vectors **e** and $\boldsymbol{\gamma}$, and another tether of length $\sqrt{2 - 2\gamma_e^+}$, connecting the ends of the vectors **e** and $-\boldsymbol{\gamma}$.

4. EXTENSION OF THE RESULTS TO THE CASE OF A HEAVY GYROSTAT, ROTATING ABOUT A FIXED POINT

The results obtained above, which touch on the integrability of the equations of motion of a heavy rigid body under unilateral constraint (2.1), can be extended to the case when, instead of the body, we consider a heavy gyrostat, also rotating about a fixed point.

In this case, the first term on the right-hand side of the equation of motion takes the form $(\mathbf{I}\boldsymbol{\omega} + \mathbf{k}) \times \boldsymbol{\omega}$, where $\mathbf{k} = (k_1, k_2, k_3)$ is the fixed vector of the gyrostatic moment. The energy integral and the geometrical integral, as before, have the form (1.2) and (1.4) respectively, while the area integral can be written in the form

$$\mathcal{T}_\psi = (\mathbf{I}\boldsymbol{\omega} + \mathbf{k}, \boldsymbol{\gamma}) \tag{4.1}$$

The impact equations (2.2) remain as before, and hence expression (2.7) for the quantity R also remains as before.

Assertion 2. The quantities

$$\mathcal{T}_e = (\mathbf{I}\boldsymbol{\omega} + \mathbf{k}, \mathbf{e}), \quad \mathcal{T}_\psi = (\mathbf{I}\boldsymbol{\omega} + \mathbf{k}, \boldsymbol{\gamma}) \tag{4.2}$$

are conserved during the time of the impact.

To prove this, we will represent relation (2.2) in the form

$$(\mathbf{I}\boldsymbol{\omega}' + \mathbf{k}) - (\mathbf{I}\boldsymbol{\omega} + \mathbf{k}) = R\mathbf{f}$$

Our further discussion will reduce to multiplying the left- and right-hand sides of the last equation scalarly by the vectors $\mathbf{e}$ and $\boldsymbol{\gamma}$ respectively.

Hence, in particular, the projection of the total angular momentum vector onto the vertical is conserved not only during the time of continuous motion, but also during the time of the impact, i.e. over the whole time of motion.

Discussions similar to those presented above prove that, in the case of a balanced gyrostat, the additional Volterra–Zhukovskii integral

$$\mathcal{T}_K = (\mathbf{I}\boldsymbol{\omega} + \mathbf{k}, \mathbf{I}\boldsymbol{\omega} + \mathbf{k})$$

for representation (2.2), generally speaking, is not conserved. However, the additional integral $\omega_3' = \omega_3$, which holds when the conditions $I_1 = I_2$, $c_1 = c_2 = 0$, $\mathbf{k} = (0, 0, k_3)$ are satisfied, and the vectors $\mathbf{c}$ and $\mathbf{e}$ are collinear, is conserved. The Sretenskii integral [4], which is similar to the Hess–Appel'rot integral, which has the form

$$F_\varepsilon = a_1 I_1 \omega_1 + \varepsilon a_3 I_3 \omega_3 + \delta = 0 \tag{4.3}$$

and which exists when conditions (3.2) and (3.3) are satisfied, is also conserved, and also

$$k_2 = 0, \quad \delta I_2 \Delta_{21} \Delta_{32} = \Delta_{32} k_1 - \varepsilon \Delta_{21} k_3 \tag{4.4}$$

the last of which defines the value of the constant δ.

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